Tutorial 11 : Selected problems of Assignment 11

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Recall Cauchy Criterion for improper integrablity :

Thm 1 (Prop. 2.18) Given $f : (a,b] \to \mathbb{R}$ such that

$\forall\, a < a' < b$, $f \in R[a',b]$, then f is improperly integrable

$\Leftrightarrow$ $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that for all $a < a' < a'' < a+\delta$,

$$\left| \int_{a'}^{a''} f \right| < \varepsilon$$

Thm 2 (Prop. 2.19) Given $f : [c, +\infty) \to \mathbb{R}$ such that

$\forall\, c' > c$, $f \in R[c,c']$, then f is improperly integrable

$\Leftrightarrow$ $\forall\, \varepsilon > 0$, $\exists\, M > c$ such that for all $c'' > c' > M$,

$$\left| \int_{c'}^{c''} f \right| < \varepsilon$$

Q1) (Supp. Q3c)

Study the improper integrability of $f(x) := \frac{\sin x}{e^x - 1}$

(a) Over $(0, 1]$

(b) Over $[1, +\infty)$

(c) Over $(0, +\infty)$

Sol^{n}: (a) We claim that f is improperly integrable over $(0,1]$.

Try to apply Theorem 1: given $\varepsilon > 0$, choose $\delta = \min\{\varepsilon, 1\}$

then for all $0 < a' < a'' < \delta$, for all $x \in [a', a'']$,

$$\begin{cases} |\sin x| \leq x \\ e^x - 1 \geq x \quad \Rightarrow \quad \frac{1}{e^x - 1} \leq \frac{1}{x} \end{cases}$$

$$\therefore \left| \int_{a'}^{a''} \frac{\sin x}{e^x - 1} \, dx \right| \leq \int_{a'}^{a''} \frac{x}{x} \, dx = (a'' - a') < \delta = \varepsilon$$

by Theorem 1, f is improperly integrable over $(0,1]$.

(b) We claim that f is improperly integrable over $[1,+\infty)$

Try to apply Theorem 2 : given $\varepsilon>0$, choose $M=\max\{\frac{2}{\varepsilon},1\}$

then for all $c''>c'\geq M$, for all $x\in[c',c'']$

$$\begin{cases} |\sin x|\leq 1 \\ e^x-1\geq \frac{x^2}{2!} \Rightarrow \frac{1}{e^x-1}\leq\frac{2}{x^2} \end{cases}$$

$$\therefore \left|\int_{c'}^{c''}\frac{\sin x}{e^x-1}\,dx\right| \leq \int_{c'}^{c''} 1\cdot\frac{2}{x^2}\,dx = \left[-\frac{2}{x}\right]_{c'}^{c''}$$

$$= 2\left(\frac{1}{c'}-\frac{1}{c''}\right) < \frac{2}{M} \leq \varepsilon$$

by Theorem 2, f is improperly integrable over $[1,+\infty)$

(c) f is improperly integrable over $(0,+\infty)$

$\overset{\text{def}}{\Leftrightarrow}$ There exists $c\in(0,+\infty)$ such that

f is improperly integrable over $(0,c]$ and $[c,+\infty)$

$\therefore$ By (a) and (b), choose $c=1 \Rightarrow f$ is improperly

Integrable over $(0,+\infty)$.